

# Zeta Functions of $\mathbb{F}_1$ -buildings

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## Introduction

It has been observed by several authors that many formulae in the geometry of Bruhat-Tits buildings over a non-archimedean field of residue cardinality  $q$  do still make sense for  $q = 1$ , in which case they coincide with the analogous formulae on the corresponding spherical geometry. Jacques Tits asked in [Tit57], whether the explanation of this phenomenon might be the existence of a “field of one element”  $\mathbb{F}_1$  such that for a Chevalley group  $G$  the group  $G(\mathbb{F}_1)$  equals the Weyl group of  $G$ . In the first decade of the new millenium, various approaches to the elusive “field”  $\mathbb{F}_1$  have been suggested, see [KOW03, Sou04, Dei05, Har07, TV09, CC10].

In the middle between the geometry of the full Bruhat-Tits building and the spherical geometry, which may be considered as the local geometry of an apartment, there is the geometry of a single apartment and its affine Weyl group. If the field of one element is the analogue of the residue field of a non-archimedean field, then the geometry of a single apartment should correspond to a “non-archimedean field in characteristic one”. In order to fix terms we shall write  $\mathbb{Q}_1$ . It appears that the approach via monoids of [Dei05] yields an explanation whereby  $\mathbb{Q}_1$  is the infinite cyclic group and one can describe the building of  $\mathrm{PGL}_n(\mathbb{Q}_1)$  in terms of lattices in perfect analogy to the case of a non-archimedean field.

Another strand of investigations, which is connected to  $\mathbb{F}_1$ -theory in this paper, is the theory of generalized Ihara zeta functions. The Ihara zeta

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function for a finite graph is defined as an Euler product over closed cycles in the graph. It turns out to be a polynomial and can be expressed in terms of the characteristic function of the adjacency operator, this latter fact being known under the name *Ihara formula*. For higher dimensional buildings, the question for a generalized Ihara formula is still open. For the case of the group  $\mathrm{PGL}_3$ , see [KLW10]. In Section 4, we present a formula of this type for  $\mathbb{F}_1$ -buildings.

## 1 The building of $\mathrm{PGL}_n(\mathbb{Q}_1)$

According to the philosophy of [Dei05] we will denote the trivial monoid of one element by  $\mathbb{F}_1 = \{1\}$ . Further we write  $\mathbb{Z}_1 = \{1, \tau, \tau^2, \dots\}$  for the free monoid of one generator  $\tau$  and  $\mathbb{Q}_1 = \{\dots, \tau^{-1}, 1, \tau, \dots\}$  for its quotient group. A *module* of a given monoid  $A$  is a set with an  $A$ -action. The category of  $A$ -modules contains direct sums, these turn out to be disjoint unions of modules. For a given natural number  $n$  consider the  $\mathbb{Q}_1$ -module  $V = V_n = \bigoplus_{j=1}^n \mathbb{Q}_1 = \coprod_{j=1}^n \mathbb{Q}_1$ . A *lattice* in  $V$  is a  $\mathbb{Z}_1$ -submodule  $L$  of  $V$  with the property that  $\mathbb{Q}_1 L = V$ . Two lattices  $L, L'$  are *homothetic* if there exists  $\alpha \in \mathbb{Q}_1$  with  $L' = \alpha L$ .

The group  $\mathrm{GL}_n(\mathbb{Q}_1)$  is by definition the automorphism group of the  $\mathbb{Q}_1$ -module  $V = \bigoplus_{j=1}^n \mathbb{Q}_1$ . Each such automorphism permutes the copies of  $\mathbb{Q}_1$  that make up  $V$  and multiplies the inhabitants of each copy by a scalar in  $\mathbb{Q}_1$ . The structure of this group is

$$\mathrm{GL}_n(\mathbb{Q}_1) \cong \mathbb{Q}_1^n \rtimes \mathrm{Per}(n),$$

where  $\mathrm{Per}(n)$  denotes the permutation group in  $n$  letters. Its center is the subgroup  $\mathrm{GL}_1(\mathbb{Q}_1) \cong \mathbb{Q}_1$  embedded diagonally. The group  $\mathrm{PGL}_n(\mathbb{Q}_1)$  is defined to be

$$\mathrm{PGL}_n(\mathbb{Q}_1) = \mathrm{GL}_n(\mathbb{Q}_1) / \mathrm{GL}_1(\mathbb{Q}_1).$$

One way to picture  $\mathrm{GL}_n(\mathbb{Q}_1)$  is to consider all  $n \times n$  matrices with exactly one non-zero entry in every row and column and this entry be in  $\mathbb{Q}_1$ . Then  $\mathrm{PGL}_n(\mathbb{Q}_1)$  consist of homothety classes of such matrices.

The building  $\mathcal{B}$  of  $\mathrm{PGL}_n(\mathbb{Q}_1)$  is the  $(n-1)$ -dimensional building defined as follows. The set of vertices is the set of all homothety classes of lattices in  $V_n$ . For  $2 \leq k \leq n-1$ , distinct vertices  $[L_0], \dots, [L_k]$  form the vertices of a  $k$ -dimensional face if, after adjusting the order, one has representatives

satisfying  $L_0 \supset L_1 \supset \cdots \supset L_k \supset \tau L_0$ . Besides the mere geometry of being a building, this lattice description of  $\mathcal{B}$  adds more features, like the order of the vertices of a face which is determined up to a cyclic permutation. First of all, there is a standard chamber  $C_0$  given by the vertices

$$\begin{aligned} L_0 &= \langle e_1, \dots, e_{n-2}, e_{n-1}, e_n \rangle, \\ L_1 &= \langle e_1, \dots, e_{n-2}, e_{n-1}, \tau e_n \rangle, \\ L_2 &= \langle e_1, \dots, e_{n-2}, \tau e_{n-1}, \tau e_n \rangle, \\ &\vdots \\ L_n &= \langle e_1, \tau e_2, \dots, \tau e_{n-2}, \tau e_{n-1}, \tau e_n \rangle. \end{aligned}$$

Here  $e_i$  stands the element 1 in the  $i$ -th copy of  $\mathbb{Q}_i$  in  $V$ . A general lattice in  $V$  can be written as

$$L = \langle \tau^{c_1} e_1, \dots, \tau^{c_n} e_n \rangle$$

for some integers  $c_1, \dots, c_n$ . We define its type to be  $c_1 + \cdots + c_n \pmod n$ . Note that the group  $(1, \dots, 1) \rtimes \text{Per}(n)$  is the stabilizer of  $L_0$ .

Observe that under this construction the building of  $\text{PGL}_n(\mathbb{Q}_1)$  is exactly isomorphic to any apartment of the building attached to  $\text{PGL}_n(\mathbb{Q}_p)$  for a prime number  $p$  and the group  $\text{PGL}_n(\mathbb{Q}_1)$  becomes the affine Weyl group of  $\text{PGL}_n(\mathbb{Q}_p)$ .

## 2 The trace formula for $\text{PGL}_n(\mathbb{Q}_1)$

Let  $\Delta \subset \mathbb{Z}^n$  denote the subgroup spanned by the element  $(1, \dots, 1)$ . We write  $G = \text{PGL}_n(\mathbb{Q}_1) \cong (\mathbb{Z}^n / \Delta) \rtimes \text{Per}(n) = \Lambda \rtimes \text{Per}(n)$ . The permutation subgroup  $\text{Per}(n)$  by  $K$ . Let  $\Gamma \subset G$  be a subgroup of finite index. The trace formula [DE09] for the pair  $(G, \Gamma)$  says that for any  $f \in \ell^1(G)$  one has

$$\sum_{\pi \in \widehat{G}} N_\Gamma(\pi) \text{tr } \pi(f) = \sum_{[\gamma]} \#(\Gamma_\gamma \backslash G_\gamma) \mathcal{O}_\gamma(f),$$

where  $L^2(\Gamma \backslash G) = \bigoplus_{\pi \in \widehat{G}} N_\Gamma(\pi) \pi$  is the decomposition into irreducibles, the sum on the right hand side runs over all conjugacy classes  $[\gamma]$  in  $\Gamma$ , the groups  $G_\gamma$  and  $\Gamma_\gamma$  are the centralizers of  $\gamma$  in  $G$  and  $\Gamma$ , and

$$\mathcal{O}_\gamma(f) = \sum_{x \in G/G_\gamma} f(x\gamma x^{-1})$$

is the *orbital sum*.

Consider the group  $G_{\mathbb{R}} = (\mathbb{R}^n / \Delta(\mathbb{R})) \rtimes \text{Per}(n)$ . We denote the subgroup  $(\mathbb{R}^n / \Delta) \rtimes \{1\} \cong \Lambda \otimes \mathbb{R}$  by  $\Lambda_{\mathbb{R}}$ .

Let  $G_{\mathbb{R}}^+$  be the set of all  $(v, p) \in G_{\mathbb{R}}$  such that  $p(v) = v$ . Then  $G_{\mathbb{R}}^+$  is the set of all  $ak \in G_{\mathbb{R}}$  with  $a \in \Lambda_{\mathbb{R}}$ ,  $k \in K$  such that  $ak = ka$ .

**Lemma 2.1.** *The set  $G_{\mathbb{R}}^+$  is a set of representatives for  $G_{\mathbb{R}}$  modulo  $\Lambda_{\mathbb{R}}$ -conjugacy. The set  $G_{\mathbb{R}}^+$  is stable under  $K$ -conjugation.*

*Proof.* Let  $(v, p) \in G_{\mathbb{R}}$  be arbitrary and  $n = (x, 1) \in \Lambda_{\mathbb{R}}$ . Then

$$n(v, p)n^{-1} = (v + x - p(x), p).$$

Let  $\text{Eig}(p, 1)$  denote the eigenspace of  $p$  for the eigenvalue 1, i.e., the set of all  $y \in \mathbb{R}^n$  such that  $p(y) = y$ . We shall show that for every  $v \in \mathbb{R}^n$  there exists  $x \in \mathbb{R}^n$  such that  $v + x - p(x)$  lies in  $\text{Eig}(p, 1)$ . This then implies the same property modulo  $\Delta$ . Since  $p$  is an orthogonal transformation on  $\mathbb{R}^n$ , there is a  $p$ -stable direct sum decomposition  $\mathbb{R}^n = \text{Eig}(1, p) \oplus \text{Eig}(1, p)^{\perp}$  and since  $1 - p$  is injective on orthogonal space  $\text{Eig}(1, p)^{\perp}$ , it is also surjective, i.e., the space  $\text{Eig}(1, p)^{\perp}$  equals the image of  $(1 - p)$ . Hence for any  $v \in \mathbb{R}^n$  there exists  $x \in \mathbb{R}^n$  such that  $v + x - p(x)$  lies in  $\text{Eig}(p, 1)$  as claimed. Therefore the set  $G_{\mathbb{R}}^+$  contains a set of representatives of  $\Lambda_{\mathbb{R}}$ -conjugacy. Next assume  $(v, p) \in G_{\mathbb{R}}^+$  and  $a \in \Lambda_{\mathbb{R}}$  with  $a(v, p)a^{-1} \in G_{\mathbb{R}}^+$ . Then  $v \in \text{Eig}(p, 1)$  and if  $a = (x, 1)$ , then  $v + x - p(x) \in \text{Eig}(p, 1)$  as well, so that  $x - p(x) \in \text{Eig}(p, 1)$ , which implies  $x - p(x) = 0$ , so  $a(v, p)a^{-1} = (v, p)$  and  $G_{\mathbb{R}}^+$  is indeed a set of representatives.  $\square$

Next we determine the unitary dual of  $G$ . The unitary dual  $\widehat{\Lambda}$  of  $\Lambda \cong \mathbb{Z}^{n-1}$  is isomorphic to  $\mathbb{T}^{n-1}$ , where  $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$  is the circle group. There is a standard way of constructing the unitary dual of a semi-direct product out of the duals of the factors. The group  $K$  acts by conjugation on  $\Lambda$  and hence on its dual  $\widehat{\Lambda}$ . For a given  $\chi \in \widehat{\Lambda}$  let  $K_{\chi}$  denote the stabilizer of  $\chi$  in the group  $K$ . For  $(\sigma, V_{\sigma}) \in \widehat{K}_{\chi}$  define  $V_{\chi, \sigma}$  to be the space of all  $\phi \in L^2(K, V_{\sigma})$  such that  $\phi(mk) = \sigma(m)\phi(k)$  holds for all  $m \in K_{\chi}$  and all  $k \in K$ . On this space we define a unitary representation  $\pi_{\chi, \sigma}$  of  $G$  by

$$\pi_{\chi, \sigma}(n, u)\phi(k) = \chi(kun(ku)^{-1})\phi(ku).$$

**Proposition 2.2.** *The unitary dual  $\widehat{G}$  of  $G$  is the set of all representations  $\pi_{\chi, \sigma}$ , where  $\chi$  runs through a set of representatives of  $\widehat{\Lambda}/K$  and  $\sigma$  runs*

through  $\widehat{K}_\chi$ . Note that as special case we have  $\chi = 1$  in which case  $K_\chi = K$ , so  $\widehat{K}$  is in a canonical way a subset of  $\widehat{G}$ . Every  $\pi \in \widehat{G}$  is finite-dimensional.

*Proof.* The space  $V_{\chi,\sigma}$  is the space of  $L^2$ -sections of the homogeneous vector bundle on the  $K$ -orbit of  $\chi$  given by the pair  $(\chi, \sigma)$ . These make up the unitary dual of the semi-direct product  $G$ .  $\square$

The group  $K \cong \text{Per}(n)$  acts on  $\Lambda_{\mathbb{R}} \cong \mathbb{R}^n / \Delta(\mathbb{R})$  and the closed cone

$$\Lambda_{\mathbb{R}}^+ = \{[x_1, \dots, x_n] \in \Lambda_{\mathbb{R}} : x_1 \geq x_2 \geq \dots \geq x_n\}$$

is a set of representatives of  $\Lambda_{\mathbb{R}}/K$ . Define elements of the dual space,

$$\alpha_1(x) = n!(x_1 - x_2), \dots, \alpha_{n-1}(x) = n!(x_{n-1} - x_n).$$

Then  $\Lambda_{\mathbb{R}}^+$  is the set of all  $x \in \Lambda_{\mathbb{R}}$  with  $\alpha_1(x) \geq 0, \dots, \alpha_{n-1}(x) \geq 0$ . Any subset  $S \subset \{1, \dots, n-1\}$  defines a face of the cone  $\Lambda_{\mathbb{R}}^+$  by

$$\Lambda_S^+ = \{x \in \Lambda_{\mathbb{R}}^+ : \alpha_j(x) = 0 \Leftrightarrow j \in S\}.$$

The cone  $\Lambda_{\mathbb{R}}^+$  is the disjoint union of its faces, in particular,  $\Lambda_{\emptyset}^+$  is the open interior of  $\Lambda_{\mathbb{R}}^+$  and  $\Lambda_{\{1, \dots, n-1\}}^+$  is the point 0.

**Lemma 2.3.** *The set  $G_{\mathbb{R}}^+ \cap (\Lambda_{\mathbb{R}}^+ \times K)$  contains a set of representatives for  $G_{\mathbb{R}}$  modulo conjugation. If  $(a, k), (a', k') \in G_{\mathbb{R}}^+ \cap (\Lambda_{\mathbb{R}}^+ \times K)$  are conjugate, then  $a = a'$  and  $k' = pkp^{-1}$  for some  $p \in K$  with  $p(a) = a$ .*

*Therefore, there exists unique conjugation-invariant functions  $l_1, \dots, l_{n-1}$  on  $G_{\mathbb{R}}$  such that*

$$l_j(v, k) = \alpha_j(v), \quad \text{if } (v, k) \in G_{\mathbb{R}}^+ \cap (\Lambda_{\mathbb{R}}^+ \times K).$$

*The functions  $l_1, \dots, l_{n-1}$  are integer-valued on the subgroup  $G$ .*

*Proof.* First,  $G_{\mathbb{R}}^+$  is a set of representatives with respect to  $\Lambda_{\mathbb{R}}$ -conjugation, which is  $K$ -stable. As every element of  $G_{\mathbb{R}}^+$  is  $K$ -conjugate to an element of  $\Lambda_{\mathbb{R}}^+ \times K$ , the first claim follows. Now let  $(a, k), (a', k') \in G_{\mathbb{R}}^+ \cap (\Lambda_{\mathbb{R}}^+ \times K)$  be conjugate, say  $a' \cdot k = (v, p)(a, k)(v, p)^{-1}$ . Then

$$(a', k') = (p(a) + v - pkp^{-1}(v), pkp^{-1}).$$

Since  $k(a) = a$ , it follows  $k'(p(a)) = p(a)$ , i.e.,  $p(a) \in \text{Eig}(k', 1)$ . As  $a' \in \text{Eig}(k', 1)$  we get  $v - pkp^{-1}(v) = v - k'(v) \in \text{Eig}(k', 1)$ . This can only be if

$v - k'(v) = 0$ , so  $a' = p(a)$ . But as  $a, a'$  are both in  $\Lambda_{\mathbb{R}}^+$ , it follows  $a = a'$  as claimed.

Finally, we need to show that  $l_j(x)$  is integral for  $x \in G$ . Write  $x = (v, p)$  with  $v \in \mathbb{Z}^n$ . Then there exists  $a \in \mathbb{R}^n$  such that  $v + (1 - p)(a) \in \text{Eig}(p, 1)$ . We have to show that this vector lies in  $\frac{1}{n!}\mathbb{Z}^n$ . For this write  $p$  a product of disjoint cycles to reduce to the case of  $p$  being one cycle, say  $p = (1, 2, \dots, k)$  for some  $k \leq n$ . Then if  $v = (1, 0, \dots, 0)$ , one finds  $a \in \mathbb{R}^n$  with

$$v - (1 - p)(a) = \frac{1}{k} \underbrace{(1, \dots, 1)}_{k\text{-times}}, 0, \dots, 0 \in \text{Eig}(p, 1).$$

From this the claim follows.  $\square$

For  $u \in \mathbb{C}^{n-1}$  and  $x \in G$  we write

$$u^{l(x)} = u_1^{l_1(x)} \dots u_{n-1}^{l_{n-1}(x)}.$$

for  $u \in \mathbb{C}^{n-1}$  let

$$\|u\|_{\max} = \max(|u_1|, \dots, |u_{n-1}|).$$

**Theorem 2.4** (Several variable Selberg type zeta function). *The infinite sum*

$$S_{\Gamma}(u) = \sum_{[\gamma]} \#(\Gamma_{\gamma} \backslash G_{\gamma}) u^{l(\gamma)}$$

converges locally uniformly for  $\|u\|_{\max} < 1$  to a rational function in  $u$ . There exist  $p_1, \dots, p_k \in \mathbb{T}^{n-1}$  and a polynomial  $Q(u)$  such that

$$S_{\Gamma}(u) = \frac{Q(u)}{\prod_{i=1}^k \prod_{j=1}^{n-1} (u_j - p_{i,j})}.$$

*Proof.* Let  $R \subset G$  be any set of representatives of  $G$  modulo conjugation. Define a function  $f_u$  on  $G$  by

$$f_u(x) = \begin{cases} u^{l(x)} & x \in R, \\ 0 & x \notin R. \end{cases}$$

Here we use the common convention that  $0^0 = 1$ .

**Lemma 2.5.** *If  $\|u\|_{\max} = \max(|u_1|, \dots, |u_{n-1}|) < 1$ , then  $f_u \in \ell^1(G)$ .*

*Proof.* By Lemma 2.3 it suffices to show that we have

$$\sum_{x \in \frac{1}{n!}\Lambda} |f_u(x)| < \infty,$$

where  $\frac{1}{n!}\Lambda = \frac{1}{n!}\mathbb{Z}^n/\Delta$ . Modulo the diagonal  $\Delta$ , every  $x \in \mathbb{R}^n$  can be assumed to have  $x_n = 0$ . Then the sum is

$$\sum_{\substack{x \in \mathbb{Z}^{n-1} \\ x_1 \geq x_2 \geq \dots \geq x_{n-1} \geq 0}} |u_1|^{x_1-x_2} \dots |u_{n-2}|^{x_{n-2}-x_{n-1}} |u_{n-1}|^{x_{n-1}}.$$

If all  $|u_j|$  are  $\leq q$  for some  $0 < q < 1$ , then each summand is less than  $q^{x_1}$ . So we have to show that for  $q < 1$  one has  $\sum_{k=0}^{\infty} c_k q^k < \infty$ , where  $c_k$  is the number of tuples of integers with  $k \geq x_2 \geq \dots \geq x_{n-1} \geq 0$  which is  $\leq (k+1)^{n-1}$ , whence the claim.  $\square$

We want to plug  $f_u$  into the trace formula. As either side of the trace formula is invariant under conjugation, neither depends on the choice of  $R$ . We give a special choice for our computations.

Every subset  $S \subset \{1, \dots, n-1\}$  defines a partition  $n = n_1 + \dots + n_r$  given by

$$\Lambda_S^+ = \{(t_1 e(n_1), \dots, t_r e(n_r)) \in \mathbb{R}^n / \Delta : t_1 > t_2 > \dots > t_r\},$$

where  $e(m) = (1, \dots, 1) \in \mathbb{R}^m$ . For  $x \in \Lambda_S^+$  we write  $x = [t_1, \dots, t_r]_S$  for these coordinates. We define

$$\Lambda_{S, \frac{1}{n}\mathbb{Z}}^+ = \left\{ \left[ \frac{k_1}{n_1}, \dots, \frac{k_r}{n_r} \right]_S : k_1, \dots, k_r \in \mathbb{Z} \right\}.$$

Let  $K_S$  denote the pointwise stabilizer of  $\Lambda_S^+$  in  $K$  and fix a set  $R_S \subset K_S$  of representatives of  $K_S$  modulo conjugation.

If  $E \subset F$  is an extension of groups, the  $F$ -conjugacy class of an element  $e$  of  $E$  intersected with  $E$ , may decompose into several  $E$ -conjugacy classes. In the following Lemma we show that this is not the case for the extension  $G \subset G_{\mathbb{R}}$ .

**Lemma 2.6.** *Every  $x \in G$  is in  $G_{\mathbb{R}}$  conjugate to a unique element of*

$$\bigcup_S \Lambda_{S, \frac{1}{n}\mathbb{Z}}^+ \times R_S.$$

*Every element of this set is  $G_{\mathbb{R}}$ -conjugate to an element of  $G$ . If  $x, y \in G$  are  $G_{\mathbb{R}}$ -conjugate, then they are  $G$ -conjugate.*

*Proof.* The first two statements are proven similar to Lemma 2.3. For the third suppose  $x = (a, k)$  and  $y = (a', k')$  are  $G_{\mathbb{R}}$ -conjugate, i.e., there exists  $(v, p) \in G_{\mathbb{R}}$  with  $(a', k') = (v + p(a) - pkp^{-1}(v), pkp^{-1})$ . Then  $v - k'(v)$  lies in the intersection of  $\text{Eig}(k', 1)^{\perp}$  and  $\mathbb{Z}^n$ . Writing  $p$  as product of disjoint cycles as in the proof of Lemma 2.3 one sees that there exists  $w \in \mathbb{Z}^n$  such that  $v - k'(v) = w - k'(w)$ .  $\square$

Let now  $\pi \in \widehat{G}$ . Then  $\pi$  is finite-dimensional and

$$\text{tr } \pi(f_u) = \sum_{x \in R} u^{l(x)} \text{tr } \pi(x).$$

Since the functions  $l_1, \dots, l_{n-1}$  are defined on  $G_{\mathbb{R}}$  and are conjugation-invariant, we may, in the computation assume that  $R$  is equal to the set in Lemma 2.6, although this set is not contained in  $G$ . In the expression  $\pi(x)$  for  $x \in R \setminus G$  we then replace  $x$  with any  $G_{\mathbb{R}}$ -conjugate inside  $G$ . We then compute

$$\text{tr } \pi(f_u) = \sum_S \sum_{a \in \Lambda^+_{S, \frac{1}{n}\mathbb{Z}}} u^{l(a)} \sum_{k \in R_S} \text{tr } \pi(ak).$$

Let  $V_{\pi} = V_{\pi,1} \oplus \dots \oplus V_{\pi,m}$  be the decomposition into  $\Lambda$ -eigenspaces, i.e., each  $a \in \Lambda$  acts on  $V_{\pi,j}$  as multiplication by  $\lambda_j^a$  for some character  $\Lambda \rightarrow \mathbb{T}$ ;  $a \mapsto \lambda_j^a$ , where  $\mathbb{T}$  is the circle group, i.e., the set of complex numbers of absolute value one. Then  $\text{tr } \pi(f_u)$  equals

$$\sum_S \sum_{j=1}^m \sum_{a \in \Lambda^+_{S, \frac{1}{n}\mathbb{Z}}} u^{l(a)} \lambda_j^a \underbrace{\sum_{k \in R_S} \text{tr } (\pi(k)|_{V_{\pi,j}})}_{=\mu_j}$$

**Lemma 2.7.** *Let  $V$  denote a  $\mathbb{Q}$  vector space of dimension  $r \in \mathbb{N}$ . Let  $V_{\mathbb{R}} = V \otimes \mathbb{R}$  and let  $C \subset V_{\mathbb{R}}$  be an open rational sharp cone with  $r$  sides, i.e., its closure  $\overline{C}$  does not contain a line and there exist  $\alpha_1, \dots, \alpha_r \in \text{Hom}(V, \mathbb{Q})$  such that*

$$C = \{v \in V_{\mathbb{R}} : \alpha_1(v) > 0, \dots, \alpha_r(v) > 0\}.$$

*Let  $\Sigma \subset V$  be a lattice, i.e., a finitely generated subgroup which spans  $V$ . Then there exists a finite subset  $S \subset \Sigma$  and  $a_1, \dots, a_r \in \Sigma$  such that  $C \cap \Sigma$  is the set of all  $v \in V$  of the form*

$$v = v_0 + k_1 a_1 + \dots + k_r a_r,$$

*where  $v_0 \in S$  and  $k_1, \dots, k_r \in \mathbb{N}_0$ . The vector  $v_0$  and the numbers  $k_1, \dots, k_r \in \mathbb{N}_0$  are uniquely determined by  $v$ .*



*Proof.* For  $j = 1, \dots, r$  let  $a_j \in \Sigma$  be the unique element such that  $\alpha_i(a_j) = 0$  for  $i \neq j$  and  $\alpha_j(a_j)$  is  $> 0$  and minimal. Then  $a_1, \dots, a_r$  is a basis of  $V$  inside  $\Sigma$ , hence it generates a sublattice  $\Sigma' \subset \Sigma$ . Let  $S$  be a set of representatives of  $\Sigma/\Sigma'$  which may be chosen such that each  $v_0 \in S$  lies in  $C$ , but for every  $j = 1, \dots, r$  the vector  $v_0 - a_j$  lies outside  $C$ . It is clear that every  $v$  of the form given in the lemma is in  $C \cap \Sigma$ .

For the converse, let  $v \in C \cap \Sigma$ . Then there are uniquely determined  $v_0 \in S$ ,  $k_1, \dots, k_r \in \mathbb{Z}$  such that  $v = v_0 + k_1 a_1 + \dots + k_r a_r$ . We have to show that  $k_1, \dots, k_r \geq 0$ . Assume that  $k_j < 0$ . Then

$$0 < \alpha_j(v) = \alpha_j(v_0) + k_j \alpha_j(a_j) \leq \alpha_j(v_0) - \alpha_j(a_j) = \alpha_j(v_0 - a_j)$$

and the latter is  $\leq 0$ , as  $v_0 - a_j$  lies outside  $C$ , a contradiction!  $\square$

We apply this lemma to  $V$  being the  $\mathbb{Q}$ -span of  $\Lambda \cap \Lambda_S^+$  and  $C = \Lambda_S^+$ . We find that  $\text{tr } \pi(f_u)$  equals

$$\begin{aligned} & \sum_S \sum_{j=1}^m \mu_j \sum_{a_0 \in F} \sum_{k_1, \dots, k_r=0}^{\infty} u^{l(a_0 + k_1 a_1 + \dots + k_r a_r)} \lambda_j^{a_1 + k_1 a_1 + \dots + k_r a_r} \\ &= \sum_S \sum_{j=1}^m \mu_j \sum_{a_0 \in F} u^{l(a_0)} \lambda_j^{a_0} \frac{1}{1 - \lambda_j^{a_1} u^{l(a_1)}} \cdots \frac{1}{1 - \lambda_j^{a_r} u^{l(a_r)}}, \end{aligned}$$

where in the last row all sums are finite. We have shown

**Lemma 2.8.** *For each  $\pi \in \widehat{G}$ , the map  $u \mapsto \text{tr } \pi(f_u)$ , defined for small  $u$ , is a rational function in  $u$ .*

We now finish the proof of the theorem. For  $\|u\|_{\max} < 1$  the function  $f_u$  goes into the trace formula. This in particular means that the sum

$$\sum_{[\gamma]} \#(\Gamma_\gamma \backslash G_\gamma) \mathcal{O}_\gamma(f_u) = S_\Gamma(u)$$

converges locally uniformly. As the quotient  $\Gamma \backslash G$  is finite, the space  $L^2(\Gamma \backslash G)$  is finite-dimensional, so the sum  $\sum_{\pi \in \widehat{G}} N_\Gamma(\pi) \text{tr } \pi(f_u)$  is a finite sum, i.e., the coefficient  $N_\Gamma(\pi)$  vanishes for almost all  $\pi$ . So  $S_\Gamma(u)$  is a finite sum of rational functions of the form in Lemma 2.8. As the representation  $\pi$  is unitary, the complex numbers  $\lambda_1, \dots, \lambda_m$  in the lemma are all in  $\mathbb{T}$ . The proof of the theorem is finished.  $\square$

### 3 Geometric interpretation

In this section we assume  $\Gamma$  to be torsion-free. It follows that  $\Gamma$  is the fundamental group of  $\Gamma \backslash \mathcal{B}$ , where  $\mathcal{B} \cong \mathbb{R}^{n-1}$  is the building of  $\mathrm{PGL}_n(\mathbb{Q}_1)$ . Then each conjugacy class  $[\gamma]$  gives a homotopy class of loops in  $\Gamma \backslash \mathcal{B}$ , where a loop is a continuous map  $S^1 \rightarrow \mathcal{B}$ . The euclidean structure makes  $\mathcal{B}$  and  $\Gamma \backslash \mathcal{B}$  a Riemannian manifold, where the geodesics in  $\mathcal{B}$  are straight lines.

**Lemma 3.1.** *Every loop on  $\Gamma \backslash \mathcal{B}$  is homotopic to a closed geodesic.*

*Proof.* Any loop is homotopic to the loop given by an element  $\gamma$  of  $\Gamma$ . As  $\Gamma$  is torsion-free and discrete,  $\gamma$  fixes no point in  $\mathcal{B}$ . The set  $P_\gamma = \{x \in \mathcal{B} : d(x, \gamma x) \text{ is minimal}\}$  is a union of straight lines on which  $\gamma$  acts by a translation. Each of these lines defines a closed geodesic.  $\square$

The group  $\Gamma$  is called a *translation group*, if  $\Gamma \subset \Lambda$ . Since  $\Lambda$  has finite index in  $G$ , every  $\Gamma$  contains a finite-index translation subgroup.

We say that a geodesic in  $\mathcal{B}$  or  $\Gamma \backslash \mathcal{B}$  is a *rational geodesic*, if it is contained in the 1-skeleton  $\mathcal{B}_1$  or  $(\Gamma \backslash \mathcal{B})_1$ .

**Lemma 3.2.** *If the homotopy class attached to a given class  $[\gamma]$  contains a rational geodesic then one has  $l_j(\gamma) = 0$  for all but one  $j \in \{1, \dots, n-1\}$ .*

*Conversely, if  $l_j(\gamma) = 0$  for all but one  $j \in \{1, \dots, n-1\}$ , then the homotopy class attached to some power  $\gamma^k$  of  $\gamma$  contains a rational geodesic. The minimal number  $k$  as above is  $\leq n$  and if  $\Gamma$  is a translation group, one always has  $k = 1$ .*

*Proof.* Suppose that the homotopy class given by  $[\gamma]$  contains a rational geodesic  $c$  in  $\Gamma \backslash \mathcal{B}$ . This means that there exists a lift  $\tilde{c} \in \mathcal{B}_1$  with  $\gamma \tilde{c}$  and  $\gamma$  induces a translation on  $\tilde{c}$ . Since  $\gamma$  preserves the affine structure on  $\mathcal{B}$ , we may choose the origin in a vertex on the rational geodesic  $\tilde{c}$ , or, which amounts to the same, conjugate  $\gamma$  by an element on  $\Lambda$ . Then  $\gamma$  maps  $x \in \mathcal{B}$  to  $Fx + t$ , where  $F(t) = t$ , so  $\gamma \in G_{\mathbb{R}}^+$ . Conjugating by some  $k \in K$ , we may assume  $t \in \Lambda_{\mathbb{R}}^+$ . As  $\tilde{c}$  is rational,  $t$  lies in a 1-dimensional face of  $\Lambda_{\mathbb{R}}^+$ , which is equivalent to saying  $l_j(\gamma) = 0$  for all but one  $j$ .

For the converse direction, assume  $\gamma x = F(x) + t$  with  $F(t) = t$ . Let  $k$  be the order of  $F$  on  $K = \mathrm{Per}(n)$ , then  $k$  is a divisor of  $n$  and  $\gamma^k$  is a translation. As  $t$  lies in a one-dimensional face of  $\Lambda_{\mathbb{R}}^+$ ,  $\gamma$  closes a rational geodesic.  $\square$

## 4 An Ihara type formula

The Ihara zeta function of a finite,  $q + 1$  regular graph  $X$  is defined as the infinite product

$$Z_X(u) = \prod_c (1 - u^{l(c)}),$$

where the product runs over all backtrackingless closed cycles  $c$  and  $l(c)$  denotes the length. The Ihara formula asserts that

$$Z_X(u) = \frac{\det(1 - Au + qu^2)}{(1 - u^2)^{-\chi}},$$

where  $A$  is the adjacency operator and  $\chi$  the Euler-characteristic. It has been proven in ascending order of generality in [Iha66, Has89, Sun86, Bas92]. For higher dimensional buildings, the question for a generalized Ihara formula is still open. For the  $\mathrm{PGL}_3$ -case see [KLW10].

Fix the following set of generators of  $\Lambda$ ,

$$S = \{[a_1, \dots, a_n] \in \Lambda, \max_{1 \leq i, j \leq n} \{|a_i - a_j|\} = 1\}.$$

The Cayley graph  $X$  of  $(\Lambda, S)$  is a  $(2^n - 2)$ -regular infinite graph naturally embedded into the Euclidean space  $\mathbb{R}^n$ , which happens to coincide with the 1-skeleton  $\mathcal{B}_1$  of the building attached to  $\mathrm{PGL}_n(\mathbb{Q}_1)$  of Section 1.

Now for  $a = [a_1, \dots, a_n] \in \Lambda$ , define its type  $\tau(a)$  as  $a_1 + \dots + a_n \pmod n$ . Fix a subgroup  $\Gamma$  of  $\Lambda$  of finite index  $N$ . Then the quotient of  $X$  by  $\Gamma$ , denoted by  $X_\Gamma$ , is the Cayley graph  $(\Lambda/\Gamma, S)$  of  $N$  vertices; it can be considered a subset of the  $(n-1)$ -dimensional torus  $\Lambda \otimes \mathbb{R}/\Gamma \cong \mathbb{R}^{n-1}/\Gamma$ . We shall assume that  $\Gamma$  only contains type zero elements so that the type is well-defined on  $\Lambda/\Gamma$ . For a function  $f : \Lambda/\Gamma \rightarrow \mathbb{C}$ , define

$$A_i f(g\Gamma) = \sum_{s \in S, \tau(s)=i} f(sg\Gamma),$$

call the type  $i$  adjacency operator of  $X_\Gamma$ . Note that  $A = A_1 + \dots + A_{n-1}$  is indeed the adjacency operator of the graph  $X_\Gamma$ .

The graph zeta function of  $X$  is given by

$$Z(X_\Gamma, u) = \prod_{[c]} (1 - u^{l(c)})$$

where  $[c]$  runs through all equivalence classes of primitive tailless backtrackless closed paths in  $X$ . Moreover, the infinite product  $Z(X_\Gamma, u)$ , convergent for small  $u \in \mathbb{C}$ , actually converges to a polynomial and it can be expressed as

$$Z(X_\Gamma, u) = \frac{\det(I_N - Au + (2^n - 3)u^2 I_N)}{(1 - u^2)^{\chi(X_\Gamma)}}$$

where  $I_N$  is the  $N \times N$  identity matrix and  $\chi(X_\Gamma)$  is the Euler characteristic of  $X_\Gamma$ . In this paper, we study a different kind of zeta function which encodes more information about the underlying space  $\mathbb{R}^n/\Gamma$ .

A path  $(v_0, \dots, v_n)$  on the graph  $X$  is called a *positive geodesic* if it is geodesic in  $\mathbb{R}^n$  and  $\tau(v_{i+1}) = \tau(v_i) + 1$  for  $i$ . A closed path  $c$  in  $X_\Gamma$  is a positive geodesic if its lifting in  $X$  is a positive geodesic;  $c$  is primitive if  $p$  is not equal to a shorter path repeated several times. Two closed paths in  $X_\Gamma$  are equivalent if one can be obtained by the other by changing the starting vertex. Denote the equivalence class of a closed path  $c$  by  $[c]$ . In this paper, we shall consider the following zeta function on  $X_\Gamma$

$$Z_+(u) = Z_+(X_\Gamma, u) = \prod_{[c]} (1 - u^{l(c)})$$

where  $[c]$  runs through all equivalence class of primitive positive closed geodesic in  $X$ .

First, we will show that

**Theorem 4.1.** *The infinite product  $Z_+(X_\Gamma, u)$  converges to a polynomial which can be expressed as*

$$Z_+(X_\Gamma, u) = \det(I_N - A_1 u + \dots + (-1)^{n-1} A_{n-1} u^{n-1} + (-1)^n u^n I_N).$$

**Remark.** As the Euler characteristic of the torus is zero, we can rewrite the above equation as

$$Z_+(X_\Gamma, u) = \frac{\det(I_N - A_1 u + \dots + (-1)^{n-1} A_{n-1} u^{n-1} + (-1)^n u^n I_N)}{(1 - u^n)^{\chi(\mathbb{R}^n/\Gamma)}}.$$

Now given a character  $\rho : \Lambda \rightarrow \mathbb{C}^\times$ , define the *Satake parameters* of  $\rho$  to be  $\rho_j = \rho(e_j)$ , where  $e_1, \dots, e_n$  is the standard basis of  $\mathbb{Z}^n$ . Then  $\rho_1 \cdots \rho_n = 1$  and we define the *Langlands L-function* of  $\rho$  to be

$$L(\rho, u) = \prod_{j=1}^n (1 - \rho_j u).$$

Finally, we define the  $L$ -function of  $\Lambda/\Gamma$  as

$$L(\Lambda/\Gamma, u) = \prod_{\rho \in \widehat{\Lambda/\Gamma}} L(\rho, u).$$

Note that there is no multiplicity showing up as an exponent to the factor  $L(\rho, u)$  as, since  $\Lambda/\Gamma$  is an abelian group, characters do not come with multiplicities other than one.

**Theorem 4.2.**  $Z_+(X_\Gamma, u) = L(\Lambda/\Gamma, u)$ .

*Proof of Theorem 4.1 and 4.2.* Let  $s_i$  be an element of  $S$  which has a representative in  $\mathbb{Z}^n$  with all coordinates equal to zero except the  $i$ -th coordinate equal to 1. Set  $S_0 = \{s_1, \dots, s_n\}$ , then

$$S = \left\{ \sum_{s \in S'} s : S' \text{ is a proper subset of } S_0 \right\}.$$

Given a vertex  $g + \Gamma$  in  $X_\Gamma$ , each positive geodesic has the form

$$g + \Gamma \rightarrow (s_i + g) + \Gamma \rightarrow (2s_i + g) + \Gamma \rightarrow \dots$$

and it is primitive if its length is equal to the order of  $s_i$  in  $\Lambda/\Gamma$ . Denote this order by  $m_i$ , then the contribution of such kind of positive closed geodesics is given by

$$(1 - u^{m_i})^{\frac{N}{m_i}} = \det(I_N - \lambda(s_i)).$$

Here  $\lambda$  is the regular representation of  $\Lambda/\Gamma$ . We conclude that

$$Z_+(X_\Gamma, u) = \prod_{i=1}^n \det(I_N - \lambda(s_i)u).$$

The right hand side is clearly equal to  $L(\Lambda/\Gamma, u)$ . On the other hand,

$$A_i = \sum_{S' \subset S_0, |S'|=i} \lambda \left( \sum_{s \in S'} s \right) = \sum_{S' \subset S_0, |S'|=i} \prod_{s \in S'} \lambda(s),$$

so that

$$I_N - A_1 u + \dots + (-1)^{n-1} A_{n-1} u^{n-1} + (-1)^n u^n I_N = \prod_{i=1}^n (I_N - \lambda(s_i)u),$$

which completes the proof of the two theorems.  $\square$

## 5 Comparison

We shall now compare the different types of zeta functions in the following theorem.

**Theorem 5.1.** *If  $\Gamma$  is a translation group, then*

$$S_\Gamma(x, 0, \dots, 0) = (n-1)! \frac{Z'_+}{Z_+}(x).$$

*Proof.* Let  $N$  be the index of  $\Gamma$  in  $\Lambda$ . Then  $N$  equals the number of vertices in  $\Gamma \backslash \mathcal{B}$ . As  $\Gamma \subset \Lambda$  we have for every  $\gamma \in \Gamma$  that  $G_\gamma = \Lambda \rtimes K_\gamma$  and so  $\#(\Gamma_\gamma \backslash G_\gamma) = N \#K_\gamma$ . In the sum  $S_\Gamma(u) = N \sum_{[\gamma]} \#K_\gamma u^{l(\gamma)}$  we find that if  $u = (x, 0, \dots, 0)$  there will only survive those summands with  $l_2(\gamma) = \dots = l_{n-1}(\gamma) = 0$ , i.e., those  $\gamma$  which are in  $G$  conjugate to an element of the form  $(c, 0, \dots, 0)$  for some  $c > 0$ . The  $K$ -centralizer  $K_\gamma$  of such an element is isomorphic to  $\text{Per}(n-1)$ , hence  $\#K_\gamma = (n-1)!$ . Such a  $\gamma$  closes a geodesic  $c$  in the 1-skeleton and the number of vertices in that geodesic equals  $l(\gamma_0)$ , where  $\gamma_0$  is the underlying primitive element. The union of all geodesics inside the 1-skeleton of  $\Gamma \backslash \mathcal{B}$  which are homotopic to  $c$  contains all vertices of  $\Gamma \backslash \mathcal{B}$ , hence, if  $k$  is their number, one has  $N = kl(\gamma_0)$ , or

$$S_\Gamma(x, 0, \dots, 0) = (n-1)! \sum_c l(c_0) x^{l(c)},$$

where the sum runs over all positive closed geodesics in  $\Gamma \backslash \mathcal{B}$ . On the other hand one has

$$\begin{aligned} \frac{Z'_+}{Z_+}(x) &= \left( \log \left( \prod_{c_0} (1 - x^{l(c_0)}) \right) \right)' \\ &= - \left( \sum_{c_0} \sum_{j=1}^{\infty} \frac{x^{l(c_0)j}}{j} \right)' \\ &= \sum_{c_0} l(c_0) \sum_{j=1}^{\infty} x^{l(c_0)j} = \sum_c l(c_0) x^{l(c)}. \end{aligned} \quad \square$$

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